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Some algbraically solvable three-body problems in one dimension

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Abstract. The three-body problem in one dimension with a repulsive inverse-square potential between every pair was solved by Calogero. Here, after mapping the three-body problem to that of a particle on a plane, the known results of supersymmetric quantum mechanics are used to solve this problem, as well as a number of new ones, algebraically. This general technique is applicable when the potential is separable in the radial and angular variables on the plane, and its supersymmetric partner is shape invariant. After discussing one example in detail, an exhaustive list of such exactly solvable potentials is given.

1. Introduction

A long time ago, in a classic paper (Calogero 1969), Calogero gave the complete solution of the Schrödinger equation for three particles in one dimension, interacting pairwise by two-body harmonic and inverse-square potentials. Later, Wolfes (1974) used Calogero's method to obtain analytical solutions of the same problem in the presence of an added three-body potential of a special form. Attention soon shifted to the exact solutions of the many-body problem (Sutherland 1971, Calogero 1971) and the general question of integrability. A list of solvable pair potentials for the many-body problem is given in the review by Olshanetsky and Perelomov (1981, 1983), where other references will be found. There are also quasi-exactly solvable potentials for which only part of the spectrum can be obtained algebraically (Shifman and Turbiner 1989, Shifman 1989). Recently, there has been a renewed interest in the one-dimensional many-body problem of the Calogero and the Sutherland types (Polychronakos 1992), and their applications to the physics of spin chains (Haldane 1988, Shastry 1988, Frahm 1993). In this paper, we present a general scheme for solving the entire spectrum of the three-body problem algebraically for a whole class of potentials using supersymmetric quantum mechanics. The potentials belonging to this class include the inverse-square pair potential of Calogero, and the three-body interaction of Wolfes, amongst others.

Three particles in one dimension, after the centre-of-mass motion is eliminated, have two independent degrees of freedom. This may therefore be mapped on to a one-body problem in two space dimensions, as was done by Calogero. All the potentials in the two-dimensional polar coordinates that are considered here have the property that the eigenvalue problem is uncoupled in the two coordinates. Even though the potential is non-central, this separability enables one to define the two constants of motion easily, as in the central potential problem. Obtaining an algebraic solution of the full problem further requires that the potentials in the radial and the angular variables be separately supersymmetric and shape invariant. Shape-invariant potentials in one dimension were discovered by Gendenshtein (1983). An extensive list of such potentials is available in the literature (Dutt *et al* 1988, Levai 1989, Khare *et al* 1991). By taking different combinations of shape-invariant potentials in the radial and angular variables, and expressing them in the one-dimensional three-particle coordinates, *we are able to construct fourteen different potentials for the three-body problem.* For each of these potentials, the eigenvalue spectrum may be obtained algebraically.

The plan of the paper is as follows. Bound-state algebraic solutions with particles in a harmonic potential are given in section 2, In this section, the general scheme is explained, and a list of the shape-invariant potentials that have been adapted for the three-body problem is given. To the extent that the list of the shape-invariant potentials in the literature is exhaustive, our list is complete. Scattering is discussed in section 3. The harmonic potential of section 2 may be replaced by a $1/r$-type potential, and this is discussed briefly in the final section.

2. Supersymmetry and the three-body problem

In this section, we first explain how supersymmetric shape-invariant potentials may be used to solve a three-body problem in one dimension. A specific example is given in some detail, followed by an exhaustive list of shape-invariant potentials that generate new examples of exactly solvable three-body problems.

Consider a potential $V(x_1, x_2, x_3)$ with which the three particles are interacting. This may consist of a sum of pairwise potentials, and/or three-body potentials. The first step is to define the Jacobi coordinates

$$X = \tfrac{1}{3}(x_1 + x_2 + x_3) \qquad x = \frac{(x_1 - x_2)}{\sqrt{2}} \qquad y = \frac{(x_1 + x_2 - 2x_3)}{\sqrt{6}}. \tag{1}$$

For potentials of physical interest, $V(x_1, x_2, x_3)$ is generally separable in the relative coordinates (x, y) and the centre-of-mass coordinate X. We shall only be interested in the relative motion of the particles, and assume, without loss of generality, that V does not depend on X. Following the notation of Calogero, define the polar coordinates (r, ϕ):

$$x = r\sin\phi \qquad y = r\cos\phi \qquad r^2 = \tfrac{1}{3}\left[(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2\right]. \tag{2}$$

Obviously, the variables r, ϕ have ranges $0 \leqslant r \leqslant \infty$, and $0 \leqslant \phi \leqslant 2\pi$. It is straightforward to show that

$$(x_1 - x_2) = \sqrt{2}r\sin\phi \qquad (x_2 - x_3) = \sqrt{2}r\sin(\phi + 2\pi/3) \qquad (x_3 - x_1) = \sqrt{2}r\sin(\phi + 4\pi/3). \tag{3}$$

The potential of the three-body problem in polar coordinates is denoted by $V(r, \phi)$, and the problem is formally equivalent to that of one particle on a plane. It is well known that further separation of the variables r and ϕ in the classical (or quantum) equations of motion takes place (Landau and Lifshitz 1976) for the form

$$V(r, \phi) = \tilde{U}(r) + \frac{U(\phi)}{r^2} \tag{4}$$

where $\tilde{U}(r)$ and $U(\phi)$ are arbitrary. For such a form, the problem is immediately integrable, and the two constants of motion are easily obtained. The Schrödinger equation for the eigenvalue problem is ($\hbar = 1$, $2M = 1$)

$$\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\Psi_{nl} = \left(E_{nl} - \tilde{U}(r) - \frac{U(\phi)}{r^2}\right)\Psi_{nl}\,. \tag{5}$$

On writing the eigenfunction $\Psi_{nl}(r, \phi)$ as

$$\Psi_{nl}(r,\phi) = \frac{1}{\sqrt{r}}u_{nl}(r)F_l(\phi) \tag{6}$$

a little algebra yields the radial equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \tilde{U}(r) + \frac{(B_l^2 - \frac{1}{4})}{r^2}\right)u_{nl}(r) = E_{nl}\,u_{nl}(r) \tag{7}$$

where B_l^2 is the eigenvalue of the angular equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\phi^2} + U(\phi)\right)F_l = B_l^2\,F_l(\phi)\,. \tag{8}$$

The two constants of motion are the eigenvalues E_{nl} and B_l^2. All this is very well known, and it is clear that the three-body problem has now reduced to the solution of the two uncoupled one-dimensional equations of one particle, given by (7) and (8).

At this stage, we use the well known result of supersymmetry that for shape-invariant potentials the solutions may be obtained algebraically (Infeld and Hull 1951, Gendenshtein 1983). To appreciate what is meant by shape invariance, consider a 'super potential' $W(s)$, where the variable s may stand for r or ϕ. The supersymmetric partners $V_-(s)$ and $V_+(s)$ are $(W^2 - W')$ and $(W^2 + W')$ respectively, the prime denoting a differentiation with respect to s. If the pair $V_\pm$ are of the same shape, but differ only in the parameters which appear in them, they are said to be shape invariant. For example, consider $V_\pm(s, a_0)$, where a_0 is a set of parameters. Shape invariance implies that

$$V_+(s, a_0) = V_-(s, a_1) + R(a_1) \tag{9}$$

where a_1 is an arbitrary function of a_0 ($a_1 = f(a_0)$), and the remainder $R(a_1)$ is independent of s. In such a case the energy spectrum of the Hamiltonian with the potential V_- is given by

$$E_n^{(-)}(a_0) = \sum_{k=1}^{n} R(a_k) \qquad E_0^{(-)}(a_0) = 0 \tag{10}$$

with $a_k = f^k(a_0)$, i.e. the function f applies k-times. Subsequently, it has been shown that the corresponding eigenfunctions can also be obtained algebraically (Dutt *et al* 1986), so the problem is completely solved. Before giving the list of three-body potentials that may be solved using this method, we consider one example in detail. Take

$$V(x_1, x_2, x_3) = V_C + V_1 \tag{11}$$

where

$$V_C = \tfrac{1}{8}\omega^2 \sum_{i<j} (x_i - x_j)^2 + g \sum_{i<j} (x_i - x_j)^{-2} \tag{12}$$

and

$$V_1 = \frac{\sqrt{3} f_1}{2r^2} \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)} + \text{cyclic terms} \right]. \tag{13}$$

Note that both V_C and V_1 are independent of the centre-of-mass coordinate X. Further, V_C is precisely the potential used by Calogero (1969), with $g > -\frac{1}{2}$ to avoid the collapse of the system. Transforming to polar coordinates, we obtain

$$\tilde{U}(r) = \tfrac{3}{8}\omega^2 r^2 \tag{14}$$

and

$$U(\phi) = \tfrac{9}{2} g \operatorname{cosec}^2 3\phi + \tfrac{9}{2} f_1 \cot 3\phi . \tag{15}$$

Here we have used the identities

$$\sum_{m=1}^{3} \operatorname{cosec}^2 [\phi + 2(m-1)\pi/3] = 9 \operatorname{cosec}^2 (3\phi) \tag{16}$$

$$\sum_{m=1}^{3} \cot [\phi + 2(m-1)\pi/3] = 3 \cot(3\phi) . \tag{17}$$

Both the radial and angular potentials give by (14) and (15) are known to be shape invariant, and the solutions may be written down algebraically (Dutt *et al* 1988, Levai 1989). For the radial part, the superpotential is

$$W(r) = \sqrt{\frac{3}{8}}\, \omega r - \frac{(B_l + \frac{1}{2})}{r} .$$

The energy eigenvalues are of the same form as obtained by Calogero

$$E_{nl} = \sqrt{\frac{3}{2}}\, \omega (2n + B_l + 1) \qquad n = 0, 1, 2 \ldots \quad l = 0, 1, 2 \ldots \tag{18}$$

and so are the eigenfunctions

$$u_{nl} = r^{B_l + 1/2} \exp\left[\frac{-1}{4} \sqrt{\frac{3}{2}}\, \omega r^2 \right] L_n^{B_l} \left[\frac{1}{2} \sqrt{\frac{3}{2}}\, \omega r^2 \right] \tag{19}$$

with $B_l > 0$. For the angular part $U(\phi)$ given by (15), the superpotential is of the form

$$W = -A \cot 3\phi - \frac{C}{A} \qquad (A > 0) \tag{20}$$

where C and A are parameters independent of ϕ. Then

$$V_{\mp}(\phi) = A(A \mp 3)\,\mathrm{cosec}^2 3\phi + 2C\cot 3\phi + \left(\frac{C^2}{A^2} - A^2\right).$$

In this example, $V_{\mp}$ are shape invariant because they obey (9), i.e.

$$V_+(A, C, \phi) = V_-(A+3, C, \phi) + (A+3)^2 - \frac{C^2}{(A+3)^2} + \frac{C^2}{A^2} - A^2 .$$

This in turn implies that the energy eigenvalues of V_- are given by

$$B_l^2 = (A+3l)^2 - \frac{C^2}{(A+3l)^2} - A^2 + \frac{C^2}{A^2} .$$

From this it follows that for the potential $U(\phi)$ given by (15), the eigenvalue spectrum of (8) is

$$B_l^2 = 9\left(l + a + \tfrac{1}{2}\right)^2 - \frac{9}{16}\frac{f_1^2}{(l + a + \frac{1}{2})^2} \tag{21}$$

where

$$a = \tfrac{1}{2}(1 + 2g)^{1/2} . \tag{22}$$

As in Calogero, $g > -\frac{1}{2}$ for meaningful solutions. The unnormalized ground-state wavefunction is given by

$$F_0(\phi) = \exp\left[-\int^{\phi} W\,\mathrm{d}\phi\right] = (\sin 3\phi)^{a+(1/2)} \exp\left[\frac{3}{4}\frac{f_1\phi}{a+\frac{1}{2}}\right] \tag{23}$$

valid in the range $0 \leqslant 3\phi \leqslant \pi$. For distinguishable particles, a given value of ϕ defines a specific ordering. For $0 \leqslant \phi \leqslant \pi/3$, equation (3) implies $x_1 \geqslant x_2 \geqslant x_3$, and other ranges of ϕ correspond to different orderings (Calogero 1969, Wolfes 1974). For singular repulsive potentials, crossing is not allowed, and $F_0(\phi)$ of (23) is zero outside $0 \leqslant \phi \leqslant \pi/3$. Following Calogero, the wavefunction for the other ranges may be constructed. For indistinguishable particles, similarly, symmetrized or antisymmetrized wavefunctions may be constructed and this will not be repeated here. From the table given by Levai (1989) the general solution $F_l(\phi)$ of (15) is given by ($\tilde{l} = l + a + \frac{1}{2}$)

$$F_l(\phi) = \exp\left[-\mathrm{i}\frac{\pi}{2}\tilde{l}\right](\sin 3\phi)^{\tilde{l}} \exp\left[\frac{3}{4}\frac{f_1\phi}{\tilde{l}}\right] P_l^{-\tilde{l}-(\mathrm{i}f_1/4\tilde{l}),\,-\tilde{l}+(\mathrm{i}f_1/4\tilde{l})}(\mathrm{i}\cot 3\phi) . \tag{24}$$

In the above equation, $P_n^{\alpha,\beta}$ is the Jacobi polynomial of the argument ($\mathrm{i}\cot 3\phi$). Note that by putting $f_1 = 0$ in (21) and (24), the usual Calogero solution for the inverse square + harmonic potential is obtained.

A similar technique as above may be used to solve other separable potentials of the form given by (4), which are displayed in table 1. The first column gives the list of potentials $U(\phi)$ that are solvable algebraically due to the shape invariance in the ϕ-coordinate. This list is complete, since no other shape-invariant potential in the angular variable is known. The

Table 1. Shape-invariant potentials in the angular variable. See text for details. Note that for (f) and (g), the eigenvalue B_l^2 does not depend on C. The equations defining A and C in terms of the parameters g and f_3 are given in the last column.

$U(\phi)$	$V(x_1, x_2, x_3)$	B_l^2
$(a)\ \frac{9}{2}g\mathrm{cosec}^2(3\phi)$	$g\sum_{i<j}(x_i - x_j)^{-2}$	$9(l+a+\frac{1}{2})^2,\ a = \frac{1}{2}(1+2g)^{1/2}$
$(b)\ \frac{9}{2}f\sec^2(3\phi)$	$3f[(x_1+x_2-2x_3)^{-2} + \text{cyclic}]$	$9(l+b+\frac{1}{2})^2,\ b = \frac{1}{2}(1+2f)^{1/2}$
$(c)\ \frac{9}{2}g\mathrm{cosec}^2(3\phi) + \frac{9}{2}f\sec^2(3\phi)$	$(a)+(b)$	$9(2l+a+b+1)^2$
$(d)\ \frac{9}{2}g\mathrm{cosec}^2(3\phi) + \frac{9}{2}f_1\cot(3\phi)$	$(a) + \frac{\sqrt{3}f_1}{2r^2}\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)} + \text{cyclic}\right]$	$9(l+a+\frac{1}{2})^2 - \frac{9}{16}f_1^2/(l+a+\frac{1}{2})^2$
$(e)\ \frac{9}{2}f\sec^2(3\phi) - \frac{9}{2}f_1\tan(3\phi)$	$(b) - \frac{3\sqrt{3}f_1}{2r^2}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)} + \text{cyclic}\right]$	Same as (d)
$(f)\ \frac{9}{2}g\mathrm{cosec}^2(3\phi) - \frac{9}{2}f_3\cot(3\phi)\mathrm{cosec}(3\phi)$	$(a) - \frac{f_3}{\sqrt{6}r}\left[\frac{(x_1+x_2-2x_3)}{(x_1-x_2)^2} + \text{cyclic}\right]$	$(A+3l)^2;\ \frac{9}{2}g = A(A-3) + C^2,$ $\frac{9}{2}f_3 = C(2A-3)$
$(g)\ \frac{9}{2}f\sec^2(3\phi) - \frac{9}{2}f_3\tan(3\phi)\sec(3\phi)$	$(b) + \frac{f_3}{3\sqrt{2}r}\left[\frac{(x_1-x_2)}{(x_1+x_2-2x_3)^2} + \text{cyclic}\right]$	$(A+3l)^2;\ \frac{9}{2}f = A(A-3) + C^2,$ $\frac{9}{2}f_3 = C(2A-3)$

next column gives the potentials in the coordinates (x_1, x_2, x_3) corresponding to $U(\phi)/r^2$. Two cyclic terms are always to be added in this column to make the potential symmetric. As a result U is a function of 3ϕ. The third column gives B_l^2, the eigenvalues of (8). The eigenfunctions and other details (like the superpotential $W(\phi)$) may be obtained by looking up Dutt *et al* (1988) and Levai (1989). The radial eigenvalue problem can also be solved algebraically by taking specific radial potentials. In the example discussed earlier, $\tilde{U}(r)$ was taken to be a simple harmonic potential. The other choice of a shape-invariant form for the radial part is an attractive $1/r$-type potential, as discussed briefly in section 4. Combining these two radial forms with the seven distinct angular potentials listed in the table yields, all in all, fourteen three-body potentials that may be solved algebraically.

There are other types of three-body potentials that may be constructed whose angular part is solvable, but is not of the supersymmetric shape-invariant form as before. In such examples, however, although the eigenfunctions $F_l(\phi)$ are expressible in terms of well known functions, the eigenspectrum of B_l^2 is not found to be in a simple closed form. For example, keeping in mind the identity

$$-(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = \frac{1}{\sqrt{2}} r^3 \sin 3\phi \tag{25}$$

a three-body potential of the form $(1/r^2)\sin^2 3\phi$ may be constructed. Combining this with the pair potential of Calogero that yielded $1/(r^2 \sin^2 3\phi)$, the angular part of the three-body Schrödinger equation may be written in the form

$$\frac{d^2 F_l}{d\phi^2} + \left[B_l^2 - a\cos 6\phi - \frac{b}{\sin^2 3\phi}\right] F_l = 0 . \tag{26}$$

It is not difficult to show that this may be reduced to the well known differential equation (Erdelyi 1955) for spheroidal wavefunctions (for $b > a - \frac{1}{2}$), whose eigenvalues, even though known, are not expressible in a simple closed form.

Until now, we have only been discussing those cases where the Schrödinger equation in the variables r and ϕ are separable, and exactly solvable. The three-body potential given by (25) is of special interest, however, even though its form does not allow separation of the coordinates r and ϕ. To be specific, consider the potential

$$V_H = \tfrac{1}{8}\omega^2 \sum_{i<j} (x_i - x_j)^2 - \lambda (x_i - x_2)(x_2 - x_3)(x_3 - x_1) . \tag{27}$$

Transforming to polar coordinates using (3), this reduces to

$$V_H = \frac{3}{8}\omega^2 r^2 + \frac{\lambda}{\sqrt{2}} r^3 \sin 3\phi \tag{28}$$

which is the famous Henon–Heiles potential (Henon and Heiles 1964) for a particle in two space dimensions. This potential is non-integrable, and has been studied extensively in connection with chaos. Therefore V_H given by (27) may be regarded as an example of a potential that gives rise to classical chaotic motion of three particles in one spatial dimension. Since it is well known that classical periodic orbits have a close connection with the quantum density of states (Gutzwiller 1990), it will be of interest to map the two-dimensional periodic orbits of the Henon–Heiles potential (Brack *et al* 1993) onto the one-dimensional motion of the three particles.

3. The three-body scattering problem

The three-body scattering problem with the inverse-square pair potential, $g \sum_{i<j} (x_i - x_j)^{-2}$, was solved by Marchioro (1969). When an additional three-body potential (see case (*c*) of table 1) is added to this, the problem is still solvable (Calogero and Marchioro 1974). Both the classical and quantum solutions are strikingly simple in these problems. Asymptotically, with the inverse-square pair potential, the momenta of the scattered particles, p_i' are related to the incident ones p_i by $p_i' = p_{4-i}$, $i = 1, 2, 3$, and the total phase shift is independent of l. With the addition of the three-body potential, the asymptotic momenta just change sign, and the phase shift still remains l-independent. We have also studied the scattering problem with the new potentials (cited in the last section) after dropping the harmonic potential. Even though the problems are still integrable, we find that the simplicity of the Calogero–Marchioro examples is lost. The scatterings with the new interactions cannot simply be expressed as an exchange between the incoming and asymptotic outgoing momenta. As an example, consider the classical scattering of three particles with the potential

$$\tilde{V}_3 = g \sum_{i<j} (x_i - x_j)^{-2} - \frac{1}{\sqrt{6}} \frac{f_3}{r} \left[\frac{(x_1 + x_2 - 2x_3)}{(x_1 - x_2)^2} + \text{cyclic terms} \right] \tag{29}$$

which is the same as given by case (*f*) in table 1. Because the variables r, and ϕ are separable, we may write, following Marchioro's notation (Marchioro 1969),

$$\tfrac{1}{2} p_r^2 + \frac{B^2}{r^2} = E \tag{30}$$

$$\tfrac{1}{2} p_\phi^2 + \tfrac{9}{2} g \operatorname{cosec}^2 3\phi - \tfrac{9}{r} 2 f_3 \cot 3\phi \operatorname{cosec} 3\phi = B^2 . \tag{31}$$

Here E is the total energy and B the angular constant of motion. It is straightforward to perform the integrations. Writing

$$b = \frac{9 f_3}{2 B^2} \qquad c = \frac{18 g}{B^2} \tag{32}$$

we obtain

$$r(t) = \left[2 E (t - t_0)^2 + \frac{B^2}{E} \right]^{1/2} \tag{33}$$

and

$$\cos 3\phi = \frac{b}{2} - \frac{1}{2} \sqrt{4 - c + b^2} \sin \left[\sin^{-1} \left(\frac{b - 2\cos 3\phi_0}{\sqrt{4 - c + b^2}} \right) + 3 \tan^{-1} \left(\sqrt{2} \frac{E}{B} (t - t_0) \right) \right] . \tag{34}$$

In the above, $\phi = \phi_0$ at $t = t_0$. Let $\phi = \phi_\mathrm{i}$ for $(t - t_0) \to \infty$, and $\phi = \phi_\mathrm{f}$ for $(t - t_0) \to -\infty$. Then

$$\cos 3\phi_\mathrm{i} = \frac{b}{2} - \frac{1}{2} \sqrt{4 - c + b^2} \cos \left[\sin^{-1} \left(\frac{b - 2\cos 3\phi_0}{\sqrt{4 - c + b^2}} \right) \right]$$

and

$$\cos 3\phi_\mathrm{f} = \frac{b}{2} + \frac{1}{2} \sqrt{4 - c + b^2} \cos \left[\sin^{-1} \left(\frac{b - 2\cos 3\phi_0}{\sqrt{4 - c + b^2}} \right) \right] . \tag{35}$$

Therefore only for $b = 0$, i.e. for $f_3 = 0$, does $\cos 3\phi_i = -\cos 3\phi_f$, leading to $3\phi_i = (\pm 3\phi_f + \pi)$. If one considers the sector for which $0 \leqslant \phi \leqslant \pi/3$, then $\phi_i = -\phi_f + \pi/3$ for $b = 0$. It was this peculiarity that yielded the simple relationships between the initial and the final asymptotic momenta of the particles. But for $f_3 \neq 0$, no such relationship exists, and the simplicity in the scattering process is lost. For the same example, similar complications arise in the quantum treatment of the scattering problem with $f_3 \neq 0$. In this case the angular wavefunction given by (34) does not possess a simple symmetry property for $\phi \to \phi - \pi/3$ when $\alpha \neq \beta$. For other examples like V_1(given by (3)) plus the inverse-square pair potential, the situation is even more complicated, and will not be discussed further.

4. Discussion

The shape-invariant potentials in the angular variable ϕ were combined with a harmonic potential in the radial coordinate to obtain exact solutions in section 2. This led to a discrete eigenvalue spectrum. The harmonic oscillator potential may be replaced by an attractive $(1/r)$-type interaction, giving rise to both discrete and continuous energies. Exact solutions for all the shape-invariant potentials described earlier can again be obtained algebraically. As one example, take the potential

$$V = -\frac{\sqrt{3}\,\alpha}{\sqrt{(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2}} + g\sum_{i<j}(x_i-x_j)^{-2}$$
$$= -\frac{\alpha}{r} + \frac{9}{2}\frac{g}{r^2\sin^2 3\phi}. \tag{36}$$

First consider the classical problem. Using the notation of (30) and (31), we obtain

$$\frac{2B^2}{\alpha r} = 1 + \sqrt{1+\frac{4EB^2}{\alpha^2}}\cos\left[\frac{1}{3}\cos^{-1}\left(\frac{\cos 3\phi}{\sqrt{1-9g/(2B^2)}}\right)\right]. \tag{37}$$

For $g = 0$, and $E < 0$, equation (37) reduces to the periodic orbit of an ellipse in the polar coordinates,

$$\frac{p}{r} = 1 + e\cos\phi \tag{38}$$

where $p = 2B^2/\alpha$ is half the latus rectum, and $e = \sqrt{(1-4|E|B^2/\alpha^2)}$ is the eccentricity of the ellipse (Landau and Lifshitz 1976). For $g = 0$, crossing between any pair of particles is allowed, and the periodic orbit given by (49) may be mapped on to the motion of three particles along a line. On the other hand, for $g \neq 0$, equation (37) does not reduce to a closed orbit for the bound problem. For the special case when $g \neq 0$ between one pair, and 0 between the other two pairs, (37) still reduces to the closed orbit form of (38). However, crossing between the interacting pair is not allowed, and the mapping in each sector has to be done accordingly. The Schrödinger equation with the potential V of (36) is easy to solve. Writing the wavefunction $\Psi_{nl}(r,\phi) = (1/\sqrt{r})u_{nl}(r)\,F_l(\phi)$ as before, the radial part obeys the standard Coulomb-type equation with the bound-state wavefunction

$$u_{nl}(r) = r^{B_l+1/2}\exp\left(-\sqrt{|E_{nl}|}\,r\right)L_n^{2B_l}(2r/r_0) \tag{39}$$

where $r_0 = 2(B_l + n + \frac{1}{2})/\alpha$, and $L_n^{2B_l}$ is the Laguerre polynomial. The corresponding eigenvalues are

$$E_{nl} = -\frac{\alpha^2}{4(n + B_l + \frac{1}{2})^2} \qquad n = 0, 1, 2, \ldots \quad l = 0, 1, 2 \ldots . \tag{40}$$

As before, the angular constants B_l are the eigenvalues of the equation

$$-\frac{d^2}{d\phi^2} + \frac{9}{2}\frac{g}{\sin^2 3\phi} F_l(\phi) = B_l^2 F_l(\phi) . \tag{41}$$

The unnormalized solutions F_l are expressed in terms of the Gegenbauer polynomials (Calogero 1969),

$$F_l = (\sin 3\phi)^{a+1/2} C_l^{a+1/2}(\cos 3\phi) \qquad 0 \leqslant \phi \leqslant \frac{\pi}{3} . \tag{42}$$

Here the constant a is given by (22), and the wavefunction vanishes outside the range $x_1 > x_2 > x_3$. The eigenvalues B_l^2 are the same as in (21) with $f_1 = 0$, so

$$B_l = 3\left(l + a + \tfrac{1}{2}\right) . \tag{43}$$

Similarly, for the continuum states, the scattering problem for the radial Coulomb part may be solved in the standard fashion (Landau and Lifshitz 1976), with the phase shift δ_l given by the usual Coulomb expression, but with l replaced by B_l.

The three-body problem in higher dimensions may also be mapped onto a one-body problem. For example, the three-body problem in two space dimensions has been reduced to the four-dimensional hyperspherical coordinates (Kilpatrick and Larsen 1987) of one particle. This method has been used to obtain the spectrum of three anyons in an oscillator potential (Khare and McCabe 1991, Law *et al* 1992). The technique of supersymmetric quantum mechanics, so aptly adapted for the one-dimensional three-body problem, has not been generalized for higher dimensions, and hence algebraic solutions cannot be obtained in such cases.

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